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LETTER TO THE EDITOR

On Einstein causality and time asymmetry in quantum physics

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Abstract

A theorem of Hegerfeldt shows that if the spectrum of the Hamiltonian is bounded from below, then the propagation speed of certain probabilities does not have an upper bound. We prove a theorem analogous to Hegerfeldt's that appertains to asymmetric time evolutions given by a semigroup of operators. As an application, we consider a characterization of relativistic quasistable states by irreducible representations of the causal Poincaré semigroup and study the implications of the new theorem for this special case.

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1. Introduction

Hegerfeldt discovered some interesting features of the structure of quantum physics appertaining to microphysical causality [1, 2]. In particular, he showed that an initially localized particle immediately develops 'infinite tails', unless it remains localized for all times. This is a very broad and general result in that it does not depend on the details specific to a particular quantum system such as the form of the Lagrangian and type of interaction. Only the existence of a positive selfadjoint Hamiltonian, and therewith the symmetry of time translations, is assumed.

In its final form, the discovery of Hegerfeldt is encapsulated in the following theorem:

Theorem 1.1 [2]. *Let H be a selfadjoint operator, positive or bounded from below, in a Hilbert space $\mathcal{H}$. For given $\psi_0 \in \mathcal{H}$, let ψ_t, $t \in \mathbb{R}$, be defined as $\psi_t = e^{-iHt}\psi_0$. Let A be a positive operator*[1] *in $\mathcal{H}$ and $p_A(t)$ be defined as $p_A(t) = \langle\psi_t, A\psi_t\rangle$. Then, either $p_A(t) \neq 0$ for almost all t and the set of such t is open and dense, or $p_A(t) = 0$ for all t.*

[1] It seems to us that Hegerfeldt's proof holds only when A is a *bounded* positive operator, though this is not explicitly stated in [2].

It is a classical result due to Stone that every selfadjoint operator H leads to a strongly continuous one-parameter group of unitary operators $U(t) = \mathrm{e}^{-\mathrm{i}Ht}$. If the spectrum of the operator H is bounded from below, then the mapping $t \to U(t)$ admits an analytic extension into the open lower half of the complex plane, $\mathbb{C}^-$, i.e., the mapping $z \to U(z) = \mathrm{e}^{-\mathrm{i}Hz}$ is strongly analytic for every z with $\Im z < 0$. Further, since $z_1 + z_2 \to U(z_1 + z_2) = U(z_1)U(z_2)$ and $\|U(z)\| \leqslant 1$, the analytic mapping $z \to U(z)$ furnishes a representation of the semigroup $\mathbb{C}^-$ (under addition) by contractions in $\mathcal{H}$. The proof of theorem (1.1) is anchored in this analytic extension of $\mathrm{e}^{-\mathrm{i}Ht}$ into the semigroup $\mathrm{e}^{-\mathrm{i}Hz}$, $z \in \mathbb{C}^-$.

In the next section, we shall show that a result analogous to Hegerfeldt's theorem holds for certain quantum systems, the time evolution of which is given by a semigroup of bounded normal operators. Specifically, we shall prove that $p_A(t) = \langle \psi_t, A\psi_t \rangle \neq 0$ for almost all $t \geqslant 0$, unless $p_A(t) \equiv 0$ for all $t \geqslant 0$. The condition $t \geqslant 0$, rather than $t \in \mathbb{R}$ as in Hegerfeldt's result, is a consequence of the fact that the time evolution is now furnished by a semigroup—not a unitary group as required for theorem 1.1.

Semigroup time evolutions are generally understood as representing a time asymmetry or an irreversibility at the quantum physical level. The main theoretical question given rise to by our mathematical result is whether such asymmetry in time translations implies or is consistent with Einstein causality. The immediate emergence of 'infinite tails' for an initially localized particle, a prediction of Hegerfeldt's theorem, remains a feature of the particular semigroup time evolution developed in the next section. Thus, insofar as particle localization is concerned, *if* such localization for a relativistic particle is meaningful in the first place, it appears that asymmetry in time evolution and causality are quite distinct notions—in particular, the former does not ensure the latter.

However, to further examine the issues of Einstein causality for irreversible processes, we must also consider the space translations consistent with the semigroup time evolution (i.e., the set of spacetime translations resulting from all possible boosts of the time evolution semigroup). The characterization of relativistic quasistable states by irreducible representations of a particular subsemigroup of the Poincaré group, studied in section 3, provides a concrete example with such a semigroup of spacetime translations. Within the context of this example, we discuss aspects of Einstein causality inferred by the main mathematical result of this paper.

2. A variant of Hegerfeldt's theorem

The following variant of theorem 1.1 holds:

Theorem 2.1. *Let $\mathcal{H}$ be a Hilbert space, and H, a normal operator in $\mathcal{H}$. Suppose the following conditions hold on $\sigma(H)$, the spectrum of H:*

$$\sup_{\lambda \in \sigma(H)} (\lambda_y) = k_0 < \infty \tag{2.1}$$

$$\sup_{\lambda \in \sigma(H), \lambda_x \leqslant 0} \left(\frac{-\lambda_y}{\lambda_x} \right) = k_1 < 0 \tag{2.2}$$

$$\inf_{\lambda \in \sigma(H), \lambda_x \geqslant 0} \left(\frac{-\lambda_y}{\lambda_x} \right) = k_2 > 0 \tag{2.3}$$

where $\lambda = \lambda_x + \mathrm{i}\lambda_y$.

Let $\mathrm{e}^{-\mathrm{i}Ht}$, $t \in [0, \infty)$, be the semigroup generated by $-\mathrm{i}H$, and let $p_A(t)$ be defined as

$$p_A(t) = \langle \psi_t, A\psi_t \rangle \tag{2.4}$$

where A is a bounded selfadjoint operator in $\mathcal{H}$, and $\psi_t = \mathrm{e}^{-\mathrm{i}Ht}\psi_0$ for given $\psi_0 \in \mathcal{H}$. Then, either

$$p_A(t) \neq 0 \text{ for almost all } t \in [0, \infty) \tag{2.5}$$

and such t are open and dense in $(0, \infty)$
or

$$p_A(t) = 0 \text{ for all } t \in [0, \infty). \tag{2.6}$$

Proof. It is well known [4] that if $\sigma(H)$ satisfies (2.1), the operator $(-\mathrm{i}H)$ generates a strongly continuous one-parameter semigroup of bounded normal operators $\mathrm{e}^{-\mathrm{i}Ht}$, $t \in (0, \infty)$. Conditions (2.2) and (2.3) ensure that the strongly continuous mapping $t \to \mathrm{e}^{-\mathrm{i}Ht}$ admits an extension into the domain $D = \{z = t + \mathrm{i}y : t > 0, k_1 < y < k_2\}$ such that it is strongly continuous on D and strongly analytic on $D\backslash(0, \infty)$.

The dual semigroup $\mathrm{e}^{\mathrm{i}H^\dagger t}$ admits an extension into $D' = \{z = t + \mathrm{i}y : t > 0, -k_2 < y < -k_1\}$. As before, the extension is continuous on D' and analytic on $D'\backslash(0, \infty)$ in the strong operator topology. Therefore, the function $p_A(t) = \langle\psi_t, A\psi_t\rangle = \langle\psi_0, \mathrm{e}^{\mathrm{i}H^\dagger t} A\, \mathrm{e}^{-\mathrm{i}Ht}\psi_0\rangle$ has an extension into $D \cap D' = \{z = t + \mathrm{i}y : t > 0, -k < y < k\}$, where $k = \min\{|k_1|, k_2\}$.

This extension, which we denote by p_A, is continuous on $D \cap D'$ and analytic on $(D \cap D')\backslash(0, \infty)$. Further, since $\overline{p_A(\bar{z})} = p_A(z)$ for every $z \in (D \cap D')\backslash(0, \infty)$ and $p_A(t)$ is real for $t \in (0, \infty)$, by the Schwarz principle of reflection, p_A is analytic in $(D \cap D')$. Since $(0, \infty)$ is in the domain of analyticity of p_A, either (2.5) or (2.6) must hold. □

Remark. Note that unlike in theorem 1.1, the positivity of A is not necessary for our proof.

3. Application to relativistic quasistable states

Wigner's pioneering work established that a correspondence exists between the unitary irreducible representations of the Poincaré group $\mathcal{P}$ and relativistic (stable) particles by way of their mass and spin. In the Hilbert space $L^2_{mj}(\mathbb{R}^3)$ of momentum wavefunctions for a particle of mass m and spin j, the relevant unitary representation can be realized by

$$(U(\Lambda, a)f)(\vec{p}, j_3) = \mathrm{e}^{-\mathrm{i}p_\mu a^\mu} \sum_{j_3'} D^j_{j_3' j_3}(W(\Lambda, p)) f(\overrightarrow{\Lambda^{-1} p}, j_3') \tag{3.1}$$

where $(\Lambda, a) \in \mathcal{P}$, $\{\vec{p}\} = \mathbb{R}^3$ and $p^2 = m^2$. The $W(\Lambda, p)$ are the well-known Wigner rotations. The D^j matrices provide a $(2j + 1)$-dimensional representation of the quantum-mechanical rotation subgroup.

The unitary irreducible representation for the particle of mass m and spin j can be realized also in the Hilbert space $L^2_j(\mathbb{R}^3)$ of 'four-velocity wavefunctions':

$$(U(\Lambda, a)f)(\vec{u}, j_3) = \mathrm{e}^{-\mathrm{i}mu_\mu a^\mu} \sum_{j_3'} D^j_{j_3' j_3}(W(\Lambda, u)) f(\overrightarrow{\Lambda^{-1} u}, j_3') \tag{3.2}$$

where $u_\mu = \frac{p_\mu}{m}$. The crucial property that makes (3.2) possible is that Wigner rotations $W(\Lambda, p)$ are functions only of the quotients $\frac{p_\mu}{m} = u_\mu$, but not of the p_μ themselves [3]. That is, $W(\Lambda, p) = W(\Lambda, u)$. The inner product under which the velocity wavefunctions form the Hilbert space $L^2_j(\mathbb{R}^3)$ is

$$(f, g) = \sum_{j_3} \int \frac{\mathrm{d}^3\vec{u}}{2u_0} \overline{f(\vec{u}, j_3)} g(\vec{u}, j_3) \tag{3.3}$$

where $\{\vec{u}\} = \mathbb{R}^3$ and $j_3 = -j, -j + 1, \ldots, j$.

It is generally believed that relativistic resonances and unstable particles are associated with simple poles of an analytic S-matrix. The location of the pole, $\mathsf{s}_R = (M - \mathrm{i}\Gamma/2)^2$, contains information about the 'mass' and 'width' of the particular quasistable state[2] while the specific partial wave in which the quasistable state occurs as a pole defines its spin j. Much like the unitary irreducible representations (3.2) of $\mathcal{P}$ which characterize isolated stable particles, we may define a representation for a quasistable state of (complex) square mass s_R and spin j in the Hilbert space $L^2_j(\mathbb{R}^3)$ by

$$(U(\Lambda, a)f)(\vec{u}, j_3) = \mathrm{e}^{-\mathrm{i}\sqrt{\mathsf{s}_R}u_\mu a^\mu} \sum_{j_3'} D^j_{j_3' j_3}(W(\Lambda, u)) f(\Lambda^{\vec{-1}}u, j_3') \tag{3.4}$$

where, as in (3.2), $\{\vec{u}\} = \mathbb{R}^3$ and $j_3 = -j, -j+1, \ldots, j$.

Note that the operators $U(\Lambda, a)$ defined by (3.4) are bounded in $L^2_j(\mathbb{R}^3)$ if and only if $(\Lambda, a) \in \mathcal{P}_+$, where

$$\mathcal{P}_+ = \{(\Lambda, a) : (\Lambda, a) \in \mathcal{P}, a_0 \geqslant 0, a^2 \geqslant 0\}. \tag{3.5}$$

The subset $\mathcal{P}_+ \subset \mathcal{P}$ is a subsemigroup of $\mathcal{P}$ as it remains invariant under the product rule $(\Lambda_1, a_1)(\Lambda_2, a_2) = (\Lambda_1\Lambda_2, a_1 + \Lambda_1 a_2)$ for $\mathcal{P}$. Further, the inverse mapping $(\Lambda, a) \to (\Lambda, a)^{-1} = (\Lambda^{-1}, -\Lambda^{-1}a)$ transforms every element $(\Lambda, a) \in \mathcal{P}_+$ (except $(I, 0)$) out of $\mathcal{P}_+$. The $\mathcal{P}_+$ is the semidirect product of the group of orthochronous Lorentz transformations and the semigroup T_+ of spacetime translations into the forward light cone:

$$T_+ = \{a : a_0 \geqslant 0, a^2 \geqslant 0\}. \tag{3.6}$$

The operators defined by (3.4) provide a continuous representation of $\mathcal{P}_+$ by contractions in $L^2_j(\mathbb{R}^3)$. This representation is characterized by a real, discrete spin value j and a complex square mass value $\mathsf{s}_R = (M - \mathrm{i}\Gamma/2)^2$, where M and Γ may be interpreted as the mass and width of the quasistable state, respectively. Note further that, as in (3.2), the Lorentz subgroup of $\mathcal{P}_+$ is unitarily represented by (3.4). It must be mentioned that complex mass representations with real velocities of the Poincaré transformations have been considered in the past [5].

The Hamiltonian $H = P_0$ for (3.4) acts in $L^2_j(\mathbb{R}^3)$ as

$$(P_0 f)(\vec{u}, j_3) = \sqrt{\mathsf{s}_R}u_0 f(\vec{u}, j_3) \qquad u_0 = (1 + \vec{u}^2)^{1/2}. \tag{3.7}$$

This shows that P_0 is a normal operator whose spectrum $\sigma(P_0) = \{(M - \mathrm{i}\Gamma/2)u_0 : u_0 \in (1, \infty)\}$. Thus, $\sigma(P_0)$ satisfies the conditions demanded in theorem 2.1, and we have a semigroup of time evolution

$$(U(t)f)(\vec{u}, j_3) = \mathrm{e}^{-\mathrm{i}\sqrt{\mathsf{s}_R}u_0 t} f(\vec{u}, j_3) = \mathrm{e}^{-\mathrm{i}M(1+\vec{u}^2)^{1/2}t}\, \mathrm{e}^{-\frac{\Gamma}{2}(1+\vec{u}^2)^{1/2}t} f(\vec{u}, j_3) \qquad t \geqslant 0. \tag{3.8}$$

Now, if $f_t = \mathrm{e}^{-\mathrm{i}P_0 t} f$ for any $f \in L^2_j(\mathbb{R}^3)$ and A is any bounded selfadjoint operator in $L^2_j(\mathbb{R}^3)$, then by theorem 2.1, $p_A(t) = \langle f_t, A f_t\rangle$ is either almost never zero for $t \geqslant 0$ or identically zero. In particular, if we choose for A a projection operator $N(V)$ representing the localization of the quasistable particle in a finite volume V, a calculation can be carried out in complete analogy to that in [2] to show that there appear 'infinite tails' for any $t \geqslant 0$. That is, the asymmetric time evolution of the quasistable state does not preclude the propagation of its survival probability into all space for arbitrarily small positive times. However, unlike Hegerfeldt's theorem, ours does not yield non-zero probabilities for $t < 0$ since the time evolution is now given by a semigroup with $t \geqslant 0$. The mathematical result notwithstanding, it is important here to mention that the spatial localization of a relativistic quasistable state, as given by the above projection operator $N(V)$, may not be a physically meaningful notion. It has been argued [6] that even for a relativistic stable particle, the localization by way of the projection $N(V)$ considered in [2] does not have much physical content.

[2] We use the term 'quasistable state' to refer to resonances and decaying states collectively.

What may be relevant and more meaningful as a directly measurable quantity (unlike survival probability) is the decay probability of the quasistable state into its possible decay products. Thus, if the operator A is taken to be the projection representing the measurement of certain decay products, then (2.4) and (2.5) infer that the probability to detect decay products becomes non-zero immediately after the quasistable state is created at $t = 0$. This property, however, does not say anything about the Einstein causality for the spacetime propagation of decay probability, a problem which requires that we consider the whole semigroup (3.6) of spacetime translations. To that end, recall first that the operators $\mathrm{e}^{-\mathrm{i}P_\mu a^\mu}$ of (3.4) are bounded in $L^2_j(\mathbb{R}^3)$ only when $a \in T_+$. For such $a \in T_+$,

$$(U(a)f)(\vec{u}, j_3) = \mathrm{e}^{-\mathrm{i}\sqrt{\mathsf{s}_R}u_\mu a^\mu} f(\vec{u}, j_3) = \mathrm{e}^{-\mathrm{i}\sqrt{\mathsf{s}_R}(u_0 t - \vec{u}.\vec{x})} f(\vec{u}, j_3) \tag{3.9}$$

where $a = (t, \vec{x})$.

Consider now a certain fixed $\vec{x}$. By (3.9), the $U(a)$ is a bounded operator only when $t \geqslant |\vec{x}|$. Further, the mapping $a \to U(a)$ given by (3.9) admits an extension into $D = \left\{z = t + \mathrm{i}y : t > |\vec{x}|,\, y < \frac{\Gamma}{2M}(t - |\vec{x}|)\right\}$ such that it is strongly continuous in D and strongly analytic in $D \backslash (|\vec{x}|, \infty)$. Starting from this observation, the proof technique of theorem 2.1 can be invoked to conclude that, for any bounded operator A in $L^2_j(\mathbb{R}^3)$ and given $f \in L^2_j(\mathbb{R}^3)$, the quantity $p_A(t, \vec{x}) = \langle U(a)f, AU(a)f\rangle$ is either almost never zero for $t \geqslant |\vec{x}|$ or identically zero.

Either of these two possibilities is clearly causal, and so what remains to be examined is $p_A(t, \vec{x})$ in spacelike regions. Ideally, we expect a causal theory to predict vanishing expectation values $p_A(t, \vec{x}) = 0$ in all regions of spacetime outside the forward semi-lightcone T_+. While the theory presented here does not yield this ideal result, certain properties of the transformation operators $U(a)$ for $a \notin T_+$ motivate the plausible argument that the theory is mathematically meaningful only for $a \in T_+$. In particular, the operators $U(a)$ are unbounded for $a \notin T_+$, and for such a, the following results are true:

(i) There exist elements f of $L^2_j(\mathbb{R}^3)$ such that $\|U(a)f\|$ is infinite.

(ii) For any $f \in L^2_j(\mathbb{R}^3)$ and $\alpha > 0$, there exists some $a \notin T_+$ such that $\|U(a)f\| > \alpha$. That is, the function $a \to \|U(a)f\|$ increases without bound for any $f \in L^2_j(\mathbb{R}^3)$.

If we want to maintain the general idea that the vectors f and $U(a)f$ represent the same physical system as viewed by two different observers, then the transformations $U(a)$ for $a \notin T_+$ need to be excluded from the theory. In particular, if $U(a)$ for $a \notin T_+$ are admitted, then it implies that for every normalized state f one can find an observer, spacelike separated from the first, to whom the quantity $\|U(a)f\|$ can be arbitrarily large. That is, there is no natural way to attribute a probability interpretation for $U(a)f$ when $a \notin T_+$. Likewise, for a state f that is initially normalized, i.e., $\|f\| = 1$, the quantity $p_A(t, \vec{x}) = \langle U(a)f, AU(a)f\rangle$ cannot be interpreted as the expectation value of the observable A in an arbitrary state $f \in L^2_j(\mathbb{R}^3)$ at $a = (t, \vec{x})$, if $a \notin \mathcal{P}_+$. It may be possible[3] that finite values can be restored if the expectation values are defined as $p_A(t, \vec{x}) = \frac{\langle U(a)f, AU(a)f\rangle}{\langle U(a)f, U(a)f\rangle}$. This, however, implies that the state vector is re-normalized for each $a \notin T_+$, and how such an operation can be given an unambiguous physical meaning is not clear.

4. Concluding remarks

Theorem 1.1 proved by Hegerfeldt seems to imply, on the face of it, that if nature admits only positive (or bounded from below) energies, then its behaviour is non-causal at the

[3] This was pointed out by a referee.

microphysical level. It may be possible to resolve this conflict, as the author suggests, once field theoretic concepts such as vacuum fluctuations and weak causality are introduced. Mathematically, the centrally significant component of Hegerfeldt's result is the semi-boundedness of the Hamiltonian and the resulting unitary group which determines the time evolution of stationary quantum systems represented in a Hilbert space. It is this reversible group evolution which brings about the non-vanishing probabilities for (almost) all t, both positive and negative.

On the other hand, it has been argued that certain quantum-mechanical processes such as decay exhibit asymmetric, irreversible time evolutions [7]. Such asymmetric time evolutions can be described by semigroups. The main technical result we reported in this paper is a variant of Hegerfeldt's theorem that applies for certain asymmetric time evolutions given by semigroups. The characterization of unstable particles by the complex mass representations of the semigroup $\mathcal{P}_+$ provides a concrete example where the new theorem can be applied. The semigroup representation (3.4) characterized by s_R and j leads to non-zero probabilities $p_A(t)$ for (almost) all $t \geqslant 0$. Negative times are excluded from the prediction by virtue of the semigroup time evolution defined only for $t \geqslant 0$.

To examine the problem of Einstein causality for the spacetime evolution of the expectation values $p_A(t, \vec{x})$ of an observable A, it is necessary to consider, in addition to the time evolution semigroup, the set of space translations consistent with this semigroup (i.e., those obtained by boosting the asymmetric time translations). It was seen in section 3 that $p_A(t, \vec{x})$ may be given a clean, unambiguous interpretation only in non-spacelike regions if the spacetime translations of the quantum system are defined by (3.4). Thus, while we have not obtained Einstein causality in the stronger sense, i.e., $p_A(t, \vec{x}) = 0$ for all $a \notin T_+$, in the foregoing weaker sense, the characterization of the quasistable states by irreducible representations of the Poincaré semigroup appears to infer Einstein causality for the expectation values of decay products.

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References

[1] Hegerfeldt G C 1974 *Phys. Rev.* D **10** 3320
Hegerfeldt G C 1985 *Phys. Rev. Lett.* **54** 2395
Hegerfeldt G C 1994 *Phys. Rev. Lett.* **72** 596
[2] Hegerfeldt G C 1998 *Irreversibility and Causality* ed A Bohm *et al* (Berlin: Springer)
[3] Werle J 1965 *Preprint* ICTP, Trieste
Bohm A 1976 *Phys. Rev.* D **13** 2110
van Dam H and Biedenharn L C 1976 *Phys. Rev.* D **14** 405
Bohm A and Kaldass H 1999 *Phys. Rev.* A **60** 4606
[4] Rudin W 1991 *Functional Analysis* (New York: McGraw-Hill)
[5] Zwanziger D 1963 *Phys. Rev.* **131** 2819
Williams D 1971 *Commun. Math. Phys.* **21** 341
Simonius M 1970 *Helv. Phys. Acta* **43** 223
[6] Buchholz D and Yngvason J 1994 *Phys. Rev. Lett.* **73** 613
[7] Lee T D 1981 *Particle Physics and Introduction to Field Theory* (New York: Harwood) chapter 13